B.Sc. Part III (Hons)

5th Paper Infinite series

Infinite product (contd.)

Q. Test the convergence of the product $\prod_{1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)$.

Sol: Let $\prod_{1}^{\infty}(1+u_n)=\prod_{1}^{\infty}\left(1+\sin^2\frac{\theta}{n}\right)$

$\Rightarrow u_n=\sin^2\frac{\theta}{n}$

$\because \sin\theta=\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\cdots$ to ∞

$\Rightarrow u_n=\sin^2\frac{\theta}{n}=\left(\frac{\theta}{n}-\frac{\theta^3}{n^3\lfloor 3}+\frac{\theta^5}{n^5\lfloor 5}-\cdots \text{ to } \infty\right)^2$

Consider another series $\sum v_n$ with $v_n=\frac{1}{n^2}$.

$\Rightarrow \frac{u_n}{v_n}=\frac{\left(\frac{\theta}{n}-\frac{\theta^3}{n^3\lfloor 3}+\frac{\theta^5}{n^5\lfloor 5}-\cdots \text{ to } \infty\right)^2}{n^2}$

$\Rightarrow \frac{u_n}{v_n}=\left(\theta-\frac{\theta^3}{n^2\lfloor 3}+\frac{\theta^5}{n^4\lfloor 5}-\cdots \text{ to } \infty\right)^2$

$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n}=\theta^2$ which is finite if θ is finite.

$\Rightarrow \sum u_n$ and $\sum v_n$ behave alike.

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent

$\Rightarrow \sum u_n$ is also convergent.

$\Rightarrow \prod_{1}^{\infty} (1+u_n)$ is also convergent.

Q. Test the convergence of the series $\prod_{1}^{\infty} \left(1 - \frac{x}{n}\right)$, $x > 0$.

Soln: Let $\prod_{1}^{\infty} (1-u_n) = \prod \left(1 - \frac{x}{n}\right)$

$\Rightarrow u_n = \frac{x}{n}$

$\Rightarrow \sum u_n = \sum \frac{x}{n} = x \sum \frac{1}{n}$

But $\sum \frac{1}{n}$ is divergent

$\Rightarrow \sum u_n = x \sum \frac{1}{n}$ is also divergent.

$\Rightarrow \prod_{1}^{\infty} (1-u_n) = \prod_{1}^{\infty} \left(1 - \frac{x}{n}\right)$ also diverges.